

# ON DISTRIBUTION OF ZEROS OF HOLOMORPHIC FUNCTIONS: BLASCHKE-TYPE CONDITIONS<sup>1</sup>

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**0. Introduction.** We use the definitions and information from [1]. Our results are closely related to [2]–[22]. As usual,  $\mathbb{N} := \{1, 2, \dots\}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are the sets of real and complex numbers, resp. The symbol  $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$  denotes the Riemann sphere (the extended complex plane);  $D(z, r) := \{z' \in \mathbb{C} : |z' - z| < r\}$  for  $z \in \mathbb{C}$  and  $r > 0$ ;  $\mathbb{D} := D(0, 1)$ . Let  $D$  be a subdomain of  $\mathbb{C}_\infty \neq D$  always here. We denote by  $\text{Hol}(D)$  and  $\text{sbh}(D)$  the class of holomorphic and subharmonic functions on  $D$ , resp. The class  $\text{sbh}(D)$  contains the function  $-\infty : z \mapsto -\infty$ ,  $z \in D$ . An elementary consequence of classical Nevalinna Theorem is

**THEOREM** (with W. Blaschke condition). *Let  $M \in \text{sbh}(\mathbb{D}) \setminus \{-\infty\}$  with the Riesz measure  $\nu_M$  and a version of the Blaschke condition*

$$\int_{1/2 \leq |z| < 1} \log \frac{1}{|z|} d\nu_M(z) < +\infty \quad (\text{Bm})$$

*is fulfilled. If a non-zero function  $f \in \text{Hol}(\mathbb{D})$  such that  $\log |f| \leq M$  on  $\mathbb{D} \setminus D(0, 1/2)$  (pointwise) and  $f$  vanish on a sequence  $Z = \{z_k\}_{k=1,2,\dots} \subset \mathbb{D}$  (we write  $f(Z) = 0$ ), then the Blaschke condition*

$$\sum_{1/2 \leq |z_k| < 1} \log \frac{1}{|z_k|} < +\infty \quad (\text{Bz})$$

*also is fulfilled.* Here the test function  $z \mapsto \log \frac{1}{|z|}$ ,  $z \in \mathbb{D}$ , from (Bm)–(Bz) is positive (sub)harmonic function on  $\mathbb{D} \setminus D(0, 1/2)$ , and

$$\lim_{1 > |z| \rightarrow 1} \log \frac{1}{|z|} = 0. \quad (0)$$

For a subset  $S$  of  $\mathbb{C}_\infty$  we denote by  $\overline{S}$  and  $\partial S$  the closure of  $S$  and the boundary of  $S$  in  $\mathbb{C}_\infty$ . If  $\overline{S}$  is a compact subset of  $D$  in the topology induced from  $\mathbb{C}_\infty$ , then we write  $S \Subset D$ . For  $S \subset \mathbb{C}_\infty$  we denote by

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$\text{sbh}(S)$  the class of functions that are subharmonic on some open set containing  $S$ ;  $\text{sbh}^+(S) := \{u \in \text{sbh}(S) : u \geq 0 \text{ on } S\}$ .

**MAIN PROBLEM.** *Let  $D_0 \Subset D$  be a subdomain,  $M \in \text{sbh}(D)$  with the Riesz measure  $\nu_M$ ,  $f \in \text{Hol}(D)$ ,  $Z = \{z_k\}_{k=1,2,\dots} \subset D$ ,  $f(Z) = 0$ , and  $\log |f| \leq M$  on  $D \setminus D_0$ . Under what conditions on the function  $v : D \setminus D_0 \rightarrow [0, +\infty)$ , for  $f \not\equiv 0$ , the implication*

$$\left( \int_{D \setminus D_0} v d\nu_M < +\infty \right) \implies \left( \sum_{z_k \in D \setminus D_0} v(z_k) < +\infty \right) \quad (\Rightarrow)$$

*is true?* This gives uniqueness theorems: if a series from  $(\Rightarrow)$  diverges, the integral from  $(\Rightarrow)$  is finite, and  $\log |f| \leq M$  on  $D \setminus D_0$ , then  $f \equiv 0$ .

**OUR SOLUTION** (see Corollary 1 and Theorem 1 below). *It is sufficient that  $v \in \text{sbh}^+(D \setminus D_0)$ , and (cf. (0))*

$$\lim_{z \rightarrow \partial D} v = 0, \quad (0)$$

i. e. for each  $\varepsilon > 0$  there is  $D_\varepsilon \Subset D$  such that  $|v| < \varepsilon$  on  $D \setminus (D_\varepsilon \cup D_0)$ .

**1.  $\delta$ -subharmonic functions** [23]–[24]. For a function or a measure  $a$  we denote by  $a|_S$  the restriction of  $a$  to  $S$ .  $L^1_{\text{loc}}(D)$  denotes the set of locally integrable functions  $f$  on  $D$  with respect to the restriction  $\lambda|_D$  of Lebesgue measure  $\lambda$ . We also consider the function  $+\infty : z \mapsto +\infty$ ,  $z \in D$ . Two functions  $\pm\infty$  are  $\delta$ -subharmonic functions. Another function  $M : D \rightarrow [-\infty, +\infty]$  is called  $\delta$ -subharmonic on  $D$  if  $M \in L^1_{\text{loc}}(D)$  and

1. For any subdomain  $D_0 \Subset D$  there exists a constant  $C_0 \in [0, +\infty)$  such that for each finite function  $\phi : D \rightarrow \mathbb{R}$  with  $\text{supp } \phi \subset D_0$ ,  $\phi \in C^2(D)$ , the inequality

$$\left| \int_{D_0} M \Delta \phi d\lambda \right| \leq C_0 \max_{z \in D_0} |\phi(z)|$$

is fulfilled. Further  $\nu_M := \frac{1}{2\pi} \Delta M$  is the Riesz charge of  $M$  where  $\Delta$  is the Laplace operator acting in the sense of distribution theory. Besides, we have the Hahn–Jordan decomposition  $\nu_M := \nu_M^+ - \nu_M^-$  where  $\nu_M^+$  and  $\nu_M^-$  are called the positive and negative part of  $\nu_M$ ;  $|\nu_M| := \nu_M^+ + \nu_M^-$  is the absolute variation of  $\nu_M$ .

2. We define  $\text{dom}_M \subset D$  as the set of points  $z \in D$  such that there is  $r_z > 0$  with the properties  $D(z, r_z) \Subset D$  and

$$\int_{D(z, r_z)} \log |z' - z_0| d|\nu_M|(z') > -\infty.$$

For  $z \in \text{dom}_M$  we set

$$M(z) = \lim_{0 < r \rightarrow 0} \frac{1}{\pi r^2} \int_{D(z, r)} M d\lambda.$$

3. Here it is convenient to set  $M(z) = +\infty$  for all  $z \in D \setminus \text{dom}_M$ .

We denote by  $\delta\text{-sbh}(D)$  the class of  $\delta$ -subharmonic functions on  $D$ . Each function  $M \in \delta\text{-sbh}(D) \setminus \{\pm\infty\}$  admits a unique representation  $M = M_+ - M_-$  where  $M_+, M_- \in \text{sbh}(D)$  with the Riesz measures  $\nu_M^+, \nu_M^-$ , resp.

**2. Main results.** For a regular domain  $D \subset \mathbb{C}_\infty$  we denote by  $g_D(\cdot, z_0)$  the extended Green's function of  $D$  with pole at  $z_0 \in D$ , that is,  $g_D(z', z_0) \equiv 0$  for all  $z' \in \mathbb{C} \setminus D$ ,  $g(\cdot, z_0) \big|_{D \setminus \{z_0\}}$  is harmonic, and  $g(\cdot, z_0)$  is subharmonic on  $\mathbb{C}_\infty \setminus \{z_0\}$ .

We denote by  $\text{const}_{a_1, a_2, \dots}$  a constant depending only on  $a_1, a_2, \dots$ .

Let  $D_0, D$  are domains in  $\mathbb{C}_\infty$ , and  $\emptyset \neq D_0 \Subset D \neq \mathbb{C}_\infty$ .

Let  $b \in [0, +\infty)$ . We set  $\text{sbh}_0^+(D \setminus D_0; \leq b)$

$$:= \left\{ v \in \text{sbh}^+(D \setminus D_0) : \lim_{z \rightarrow \partial D} v(z) \stackrel{\text{see}}{=} 0, \sup_{z \in \partial D_0} v(z) \leq b \right\}.$$

We consider a function  $M \in \delta\text{-sbh}(D) \setminus \{\pm\infty\}$  with the Riesz charge  $\nu_M$ .

MAIN THEOREM. *FOR ANY*

(i) *point*  $z_0 \in D_0 \cap \text{dom}_M$  *and number*  $b \in [0, +\infty)$ ,

(ii) *regular domain*  $\tilde{D} \subset \mathbb{C}_\infty$ ,  $D_0 \Subset \tilde{D} \subset D$ ,  $\mathbb{C}_\infty \setminus \overline{\tilde{D}} \neq \emptyset$ ,

*THERE EXISTS a number*  $C := \text{const}_{z_0, D_0, \tilde{D}, D, b} > 0$ , *such that for each*  $u \in \text{sbh}(D) \setminus \{-\infty\}$  *the inequality*  $u \leq M$  *on*  $D \setminus D_0$  *entails that, for*

any function  $v \in \text{sbh}_0^+(D \setminus D_0; \leq b)$ , the inequality

$$\begin{aligned} Cu(z_0) + \int_{D \setminus D_0} v d\nu_u &\leq \int_{D \setminus D_0} v d\nu_M + \int_{\tilde{D} \setminus D_0} v d\nu_M^- \\ &+ C \int_{\tilde{D}} g_{\tilde{D}}(\cdot, z_0) d\nu_M + C \int_{\tilde{D} \setminus D_0} g_{\tilde{D}}(\cdot, z_0) d\nu_M^- + CM(z_0) \end{aligned}$$

is fulfilled. Moreover, if  $\tilde{D} \Subset D$ , then this estimate can be replaced by

$$\int_{D \setminus D_0} v d\nu_u \leq \int_{D \setminus D_0} v d\nu_M + (b + C) \overline{C}_M - Cu(z_0), \quad (\text{C})$$

where a constant  $\overline{C}_M := \text{const}_{z_0, D_0, \tilde{D}, D, M} < +\infty$  is positively homogeneous of  $M$ , i. e.  $\overline{C}_{aM} = a\overline{C}_M$  for  $a \in [0, +\infty)$ , and upper semi-additive of  $M$ , i. e.  $\overline{C}_{M_1+M_2} \leq \overline{C}_{M_1} + \overline{C}_{M_2}$  for suitable  $M_1, M_2$ .

Conversely, IF the function  $M$  is continuous on regular domain  $D$  and, for a Borel measure  $\nu \geq 0$  on  $D$  and for a number  $b > 0$ , there is a constant  $C'$  such that for each function  $v \in \text{sbh}_0^+(D \setminus D_0; \leq b)$  the inequality

$$\int_{D \setminus D_0} v d\nu \leq \int_{D \setminus D_0} v d\nu_M + C' \quad (\text{C}')$$

is fulfilled, THEN there exists a function  $u \in \text{sbh}(D) \setminus \{-\infty\}$  with the Riesz measure  $\nu_u \geq \nu$  such that  $u \leq M$  on  $D$ , and, for any function  $u_0 \in \text{sbh}(D)$  with the Riesz measure  $\nu_{u_0} = \nu$  and for any number  $\varepsilon > 0$ , there exists a non-zero function  $f \in \text{Hol}(D)$  such that the inequality

$$u_0(z) + \log|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} M(z + re^{i\theta}) d\theta + \log \frac{(1 + |z|)^{1+\varepsilon}}{r} \quad (\text{L})$$

is fulfilled for all  $z \in D$  and for all

$$0 < r < \min\left\{1 + |z|, \text{dist}(z, \partial D) := \inf\{|z' - z| : z' \in \partial D\}\right\}. \quad (\text{d})$$

**Remark.** In addition, if  $D = \mathbb{C}$  or  $D$  is a domain with smooth boundary of class  $C^1$  and, for  $v|_{\partial D} \equiv 0$ , there exists the derivative  $\frac{\partial v}{\partial \bar{n}_{\text{in}}}|_{\partial D} \equiv 0$

along the inner normal  $\vec{n}_{\text{in}}$  of  $\partial D$ , then the integral in the right parts of the inequalities (C) and (C') can be replaced by the integral  $\int_{D \setminus D_0} M d\nu_v$ ,

where  $\nu_v$  is the Riesz measure of  $v$ .

**THEOREM 1.** *Let  $M \in \text{sbh}(D)$ ,  $b \in [0, +\infty)$ . Then there are numbers  $C := \text{const}_{D_0, D, b} > 0$ ,  $\overline{C}_M := \text{const}_{D_0, D, M} \geq 0$  such that, for any  $v \in \text{sbh}_0^+(D \setminus D_0; \leq b)$  and for each  $u \in \text{sbh}(D) \setminus \{-\infty\}$ , the inequality  $u \leq M$  on  $D \setminus D_0$  entails the inequality (C).*

*Conversely, if, for a Borel measure  $\nu \geq 0$  on regular domain  $D$ , there is a constant  $C'$  such that for each function  $v \in \text{sbh}_0^+(D \setminus D_0; \leq b)$  the inequality (C') is fulfilled, then there exists  $u \in \text{sbh}(D) \setminus \{-\infty\}$  with the Riesz measure  $\nu_u \geq \nu$  such that  $u \leq M$  on  $D$ , and, for any function  $u_0 \in \text{sbh}(D)$  with the Riesz measure  $\nu_{u_0} = \nu$  and for any number  $\varepsilon > 0$ , there exists a non-zero function  $f \in \text{Hol}(D)$  satisfying (L)-(d).*

*The Remark remains in force.*

**COROLLARY 1.** *Let  $D_0 \Subset D$  be a subdomain of  $D$ ,  $M \in \text{sbh}(D)$  with the Riesz measure  $\nu_M$ ,  $f \in \text{Hol}(D)$ ,  $Z = \{z_k\}_{k=1,2,\dots} \subset D$ ,  $f(Z) = 0$ , and  $\log|f| \leq M$  on  $D \setminus D_0$ . If  $v \in \text{sbh}^+(D \setminus D_0)$ , and  $\lim_{z \rightarrow \partial D} v \stackrel{\text{see(O)}}{=} 0$ , then the implication  $(\Rightarrow)$  from Main Problem is true.*

**COROLLARY 2.** *Let  $D = \mathbb{C}$  or  $D$  be a domain with smooth boundary of class  $C^1$ , and  $D_0 \Subset D$  be a subdomain of  $D$ . Suppose that a continuous function  $v: \overline{D} \setminus D_0 \rightarrow \mathbb{R}$  satisfies the following conditions*

- $v|_{D \setminus D_0} \in \text{sbh}^+(D \setminus D_0)$  with the Riesz measure  $\nu_v$ , and  $v|_{\partial D} \equiv 0$ ,
- and there exists the derivative  $\frac{\partial v}{\partial \vec{n}_{\text{in}}}|_{\partial D} \equiv 0$ .

*If a non-zero function  $f \in \text{Hol}(D)$  vanishes on a sequence  $Z = \{z_k\} \subset D$ , and  $f$  satisfies the condition  $\int_{D \setminus D_0} \log|f| d\nu_v < +\infty$ , then*

$$\sum_{z_k \in D \setminus D_0} v(z_k) < +\infty.$$

Our results allow to obtain various forms of development and generalization of results from [2]–[22], as well as new results both for the complex plane  $\mathbb{C}$  and for domains in  $\mathbb{C}_\infty$  such as  $\mathbb{D}$ , and another domains. Detailed treatment of these and other results are submitted to the journal «Математический сборник».

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